Topological-Like Features in Diagrammatical Quantum Circuits [1]

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Abstract

In this paper, we revisit topological-like features in the extended Temperley–Lieb diagrammatical representation for quantum circuits including the teleportation, dense coding and entanglement swapping. We perform these quantum circuits and derive characteristic equations for them with the help of topological-like operations. Furthermore, we comment on known diagrammatical approaches to quantum information phenomena from the perspectives of both tensor categories and topological quantum field theories. Moreover, we remark on the proposal for categorical quantum physics and information to be described by dagger ribbon categories.

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[1]This manuscript is a formal written version of Y. Zhang's talk at the workshop "Cats, Kets and Cloisters" (Computing Laboratory, Oxford University, July 17-23, 2006), in which he has been proposing categorical quantum physics & information to be described by dagger ribbon categories and emphasizing the functor between Abramsky and Coecke's categorical quantum mechanics and his extended Temperley–Lieb categorical approach to be the same type as those defining topological quantum field theories. As a theoretical physicist, however, the proposer himself has to admit that these arguments are rather mathematical type so that they are hardly appreciated by physicists because physics is such a great field including various kinds of topics and topology only plays important roles in a limited number of physical problems in the present knowledge. On the other hand, this proposal is suggesting either that fundamental objects in the physical world are string-like (even brane-like) and satisfy the braid statistics or that quasi-particles of many-body systems (or fundamental particles at the Planck energy scale) obey the braid statistics and have an effective (or a new internal) degree of freedom called the "twist spin", so that the braiding and twist operations for defining ribbon categories would obtain a reasonable and physical interpretation. Furthermore, this name "categorical quantum physics and information" hereby refers to quantum physics and information which can be recast in terms of the language of categories, and it is a simple and intuitional generalization of the name "categorical quantum mechanics" because the latter does not recognize conformal field theories, topological quantum field theories, quantum gravity and string theories which have been already described in the categorical framework by different research groups. Moreover, the proposal *categorical quantum physics and information* has been strongly motivated by the present study in quantum information phenomena and theory, and it is aimed at setting up a theoretical platform on which both categorical quantum mechanics and topological quantum computing by Freedman, Larsen and Wang are allowed to stand.

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1 Introduction

It is well known that diagrams are capable of catching essential points from the global view so that they can express complicated algebraic objects in a much simpler style. Recently, there have been several diagrammatical approaches proposed to study quantum information phenomena [1]. Abramsky and Coecke [2, 3, 4] exploit a generalized diagrammatical representation for tensor categories as a substantial extension of Dirac's notation to describe quantum information protocols in the language of strongly compact closed categories. Kauffman and Lomonaco [5, 6] show the relationship between the teleportation procedure and the diagrammatical matrix formalism used in quantum topology and they call it the teleportation topology. Griffiths et al. [7] devise a set of atemporal diagrams without reference to time to present quantum circuits.

Furthermore, instead of strongly compact closed categories [2, 3, 4], Zhang [8, 9] proposes that the Temperley–Lieb (TL) algebra [10] under local unitary transformations underlies quantum information protocols involving maximally entangled states, projective measurements and local unitary transformations, and he names the extended TL category for a collection of all the TL algebras under local unitary transformations. Also, various descriptions for the quantum teleportation are found to have a unified description in terms of the extended TL configurations. See [8, 9] for this proposal and consult [11, 12] for an introduction to the TL algebra and the Brauer algebra (i.e., the TL algebra with permutation) [13]. Moreover, Kauffman and Lomonaco [14] relate the 0-dimensional cobordism category to the Dirac notation of bras and kets and to the quantum teleportation.

In this paper, we go further to explore "topological" [3] features in the extended TL diagrammatical representation for a quantum circuit. We define topological-like operations as continuous deformations of a diagrammatical configuration, and with the help of them, perform quantum circuits and derive characteristic equations for the teleportation [15, 16, 17, 18], dense coding and entanglement swapping [19]. Besides this, we will contribute a section for comments on known diagrammatical approaches to quantum information phenomena including diagrammatics for categorical quantum mechanics [2, 3, 4], atemporal diagrammatical representation [7], the extended TL categorical approach [8, 9] and 0-dimensional cobordism category [14], and then explain the suggestion (in the first footnote) that all of them are related to diagrammatics for tensor categories and the mapping between categorical quantum mechanics and the extended TL categorical approach (i.e., 0-dimensional cobordism category or the Brauer category) is the same type of functor as those defining topological quantum field theories (TQFT) [20]. Eventually, we make reliable reasons for the proposal (in the first footnote) that dagger ribbon categories are responsible for the description of

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[3]In this paper, we only have "topological" or topological-like deformations because topological deformations of the diagrams are only allowed if they do not change the algebraic interpretation or if they correspond to an algebra identity. Our descriptions for quantum circuits are also "topological" or topological-like because we are showing that some diagrammatical representations of quantum circuits are subject to limited deformations where the representation of the algebra by diagrams corresponds to such deformations.

categorical quantum physics & information.

The plan of this paper is organized as follows. The extended TL diagrammatical rules are revisited Section 2, examples for "topological" operations are listed Section 3, the teleportation [16, 17] and entanglement swapping [19] are performed via "topological" operations Section 4, and characteristic equations for the teleportation, dense coding and entanglement swapping are respectively derived Section 5. Comments on known diagrammatical approaches are made Section 6. Concluding remarks are on observations for categorical quantum physics & information. A brief introduction to the extended TL algebra and various categorical structures exploited in this manuscript are respectively made Appendix A and Appendix B.

2   Extended Temperley–Lieb diagrammatical rules

Maximally entangled states with interesting algebraic properties play key roles in quantum information and computation. We review extended TL diagrammatical rules [8, 9] for mapping every diagrammatical element to an algebraic term in order to describe algebraic objects in terms of maximally entangled states.

2.1   Notations for maximally entangled states

The vectors $|e_i\rangle$, $i = 0, 1, \cdots d - 1$ form a set of orthonormal bases for a d-dimension Hilbert space $\mathcal{H}$, and the covectors $\langle e_i|$ are chosen for its dual Hilbert space $\mathcal{H}^\dagger$,

$$\sum_{i=0}^{d-1} |e_i\rangle\langle e_i| = \mathbb{1}_d, \qquad \langle e_j|e_i\rangle = \delta_{ij}, \qquad i, j = 0, 1, \cdots d - 1, \tag{1}$$

where δ_{ij} is the Kronecker symbol and $\mathbb{1}_d$ denotes the $d \times d$ unit matrix. A maximally bipartite entangled state vector $|\Omega\rangle$ and its dual state vector $\langle\Omega|$ have the forms,

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i \otimes e_i\rangle, \qquad \langle\Omega| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle e_i \otimes e_i|, \tag{2}$$

The action of a bounded linear operator M in the Hilbert space $\mathcal{H}$ on $|\Omega\rangle$ satisfies

$$(M \otimes \mathbb{1}_d)|\Omega\rangle = (\mathbb{1}_d \otimes M^T)|\Omega\rangle, \qquad M^T_{ij} = M_{ji}, \qquad M_{ij} = \langle e_i|M|e_j\rangle, \tag{3}$$

where the upper index T denotes the transpose, and hence this is permitted to move the local action of the operator M from the Hilbert space to the other Hilbert space as it acts on $|\Omega\rangle$. The trace of two operators M and N can be represented by an inner product between maximally entangled state vectors,

$$tr(MN) = d \cdot \langle\Omega|(M \otimes \mathbb{1}_d)(N \otimes \mathbb{1}_d)|\Omega\rangle. \tag{4}$$

The transfer operator T_{BC}, sending a quantum state from Charlie to Bob, is recognized to be another inner product between $_{CA}\langle\Omega|$ and $|\Omega\rangle_{AB}$,

$$T_{BC} \equiv \sum_{i=0}^{d-1} |e_i\rangle_B \,_C\langle e_i|, \qquad T_{BC}|\psi\rangle_C = |\psi\rangle_B, \qquad T_{BC} = d \cdot \,_{CA}\langle\Omega|\Omega\rangle_{AB}, \tag{5}$$
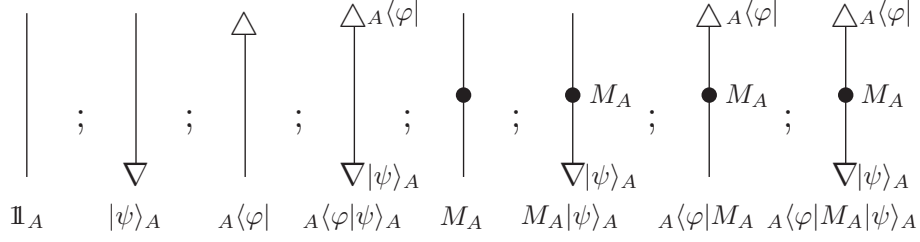
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Figure 1: Straight lines without or with points.

which has been exploited by Braunstein et al. in the mathematical description for quantum teleportation schemes [18].

The maximally entangled vector $|\Omega_n\rangle$ is a local unitary transformation of $|\Omega\rangle$, i.e., $|\Omega_n\rangle = (U_n \otimes \mathbb{1}_d)|\Omega\rangle$, and the set of unitary operators U_n satisfies the orthogonal relation $tr(U_n^\dagger U_m) = d\,\delta_{nm}$, which leads to the following properties,

$$\langle\Omega_n|\Omega_m\rangle = \delta_{nm}, \qquad \sum_{n=1}^{d^2} |\Omega_n\rangle\langle\Omega_n| = \mathbb{1}_d, \qquad n, m = 1, \cdots d^2, \tag{6}$$

where the upper index $\dagger$ denotes the adjoint. Introduce the symbol ω_n for the maximally entangled state $|\Omega_n\rangle\langle\Omega_n|$ and especially denote $|\Omega\rangle\langle\Omega|$ by ω, namely,

$$\omega \equiv |\Omega\rangle\langle\Omega|, \qquad \omega_n \equiv |\Omega_n\rangle\langle\Omega_n|, \qquad U_1 = \mathbb{1}_d, \tag{7}$$

and the set of ω_n, $n = 1, 2, \cdots d^2$ forms a set of observables over an output parameter space.

2.2 Extended TL diagrammatical rules

Three pieces of extended TL diagrammatical rules are devised for assigning a diagram to a given algebraic object. The first is our convention; the second explains what straight lines and oblique lines represent; the third describes various configurations in terms of cups and caps.

Rule 1. Read an algebraic object such as an inner product from the left-hand side to the right-hand side and draw a diagram from the top to the bottom. Represent the operator M by a solid point, its adjoint operator $M^\dagger$ by a small circle, its transposed operator M^T by a solid point with a cross line and its complex conjugation operator M^* by a small circle with a cross line. Denote the Dirac ket by the symbol ∇ and the Dirac bra by the symbol $\triangle$.

Rule 2. See Figure 1. A straight line of type A denotes the identity operator $\mathbb{1}_A$ for the system A. Straight lines of type A with a bottom ∇ or top $\triangle$ describe a vector $|\psi\rangle_A$, covector $_A\langle\varphi|$, and an inner product $_A\langle\varphi|\psi\rangle_A$ in the system A, respectively. Straight lines of type A with a middle solid point or bottom ∇ or top $\triangle$ describe an
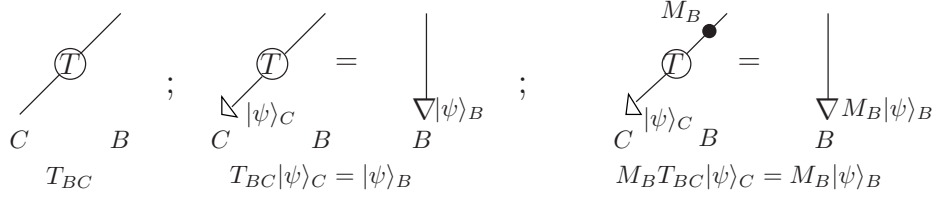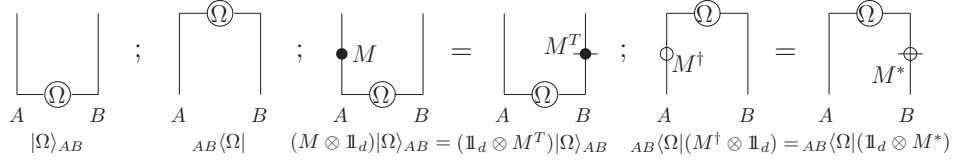
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Figure 2: Oblique line for transfer operator.

Figure 3: Cups and caps without or with points.

operator M_A, a vector $M_A|\psi\rangle_A$, a covector $_A\langle\varphi|M_A$, and an inner product $_A\langle\varphi|M_A|\psi\rangle_A$, respectively.

See Figure 2. An oblique line from the system C to the system B describes the transfer operator T_{BC}, and its solid point or bottom ∇ or top $\triangle$ have the same interpretations as those on a straight line of type A in Figure 1.

Rule 3. See Figure 3. A cup denotes the maximally bipartite entangled state vector $|\Omega\rangle$ and a cap does for its dual $\langle\Omega|$. A cup with a middle solid point on its one branch describes a local action of the operator M on $|\Omega\rangle$, and this solid point can flow to its other branch and is replaced by a solid point with a cross line representing M^T. The same happens for a cap except that a solid point is replaced by a small circle to distinguish the operator M from its adjoint operator $M^\dagger$.

A cup and a cap can form different sorts of configurations. See Figure 4. As a cup is at the top and a cap is at the bottom for the same composite system, this configuration is assigned to the projector $|\Omega\rangle\langle\Omega|$. As a cap is at the top and a cup is at the bottom for the same composite system, this diagram describes an inner product $\langle\Omega|\Omega\rangle = 1$ by a closed circle. As a cup is at the bottom for the composite system $\mathcal{H}_C \otimes \mathcal{H}_A$ and a cap is at the top for the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$, that is an oblique line representing the transfer operator T_{BC} with the normalization factor $\frac{1}{d}$.

Additionally, as a cup has a local action of the operator M and a cap has a local action of the operator $N^\dagger$, the resulted circle with a solid point for M and a small circle for $N^\dagger$ represents the trace $\frac{1}{d}tr(MN^\dagger)$. As conventions, we describe a trace of operators by a closed circle with solid points or small circles, and assign each cap or cup a normalization factor $\frac{1}{\sqrt{d}}$ and a circle a normalization factor d.

Note that cups and caps are well known configurations in knot theory and statistics

Figure 4: Three kinds of combinations of a cup and a cap.

Figure 5: Cup and cap via compositions of cups and caps.

mechanics. They were used by Wu [21] in statistical mechanics, and exploited by Kauffman [22] for diagrammatically representing the Temperely-Lieb algebra soon after Jones's work [23]. These configurations are nowadays called Brauer diagrams [13] or Kauffman diagrams [22].

3   Examples for "Topological" operations

Topological-like operations are defined as continuous deformations of diagrammatical configurations, and three typical examples exploited in what follows are presented.

See Figure 5 for two kinds of compositions of cups and caps. The configuration of a cup (cap) can be regarded as a composition of a series of cups and caps. The cup state $|\Omega\rangle_{AD}$ is obtained by connecting the cap state $_{BC}\langle\Omega|$ with the cup states $|\Omega\rangle_{AB}$ and $|\Omega\rangle_{CD}$, namely proved by

$$(_{BC}\langle\Omega|)(|\Omega\rangle_{AB})(|\Omega\rangle_{CD}) = \frac{1}{d}|\Omega\rangle_{AD}, \tag{8}$$

and the cap state $_{AD}\langle\Omega|$ is a composition of the cap states $_{AB}\langle\Omega|$, $_{CD}\langle\Omega|$ and the cup state $|\Omega\rangle_{BC}$, specified by

$$(_{AB}\langle\Omega|)(_{CD}\langle\Omega|)(|\Omega\rangle_{BC}) = \frac{1}{d}\,_{AD}\langle\Omega|. \tag{9}$$

See Figure 6 for two sorts of diagrammatical partial traces. The partial trace of a composite system denotes the summation over its subsystem, for example,

$$tr_A(|e_i^C \otimes e_j^A\rangle\langle e_l^A \otimes e_m^B|) = |e_i^C\rangle\langle e_m^B|\delta_{jl}, \qquad tr_A(|e_j^A\rangle\langle e_l^A|) = \delta_{jl}, \tag{10}$$
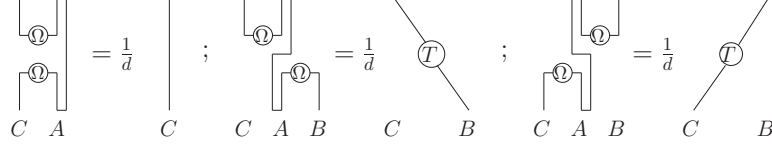
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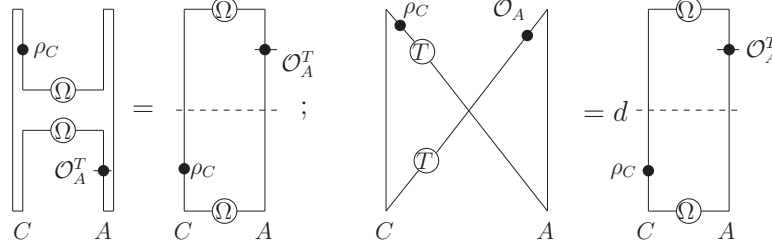Figure 6: Three kinds of partial traces between a cup and a cap.

Figure 7: Closed circles via cup and cap, and oblique lines

while the trace is defined as the summation over the entire composite system,

$$tr_{CA}(|e_i^C \otimes e_j^A\rangle\langle e_l^C \otimes e_m^A|) = \delta_{il}\delta_{jm}, \qquad i, j, l, m = 0, 1, \cdots d - 1. \tag{11}$$

The first type of diagrammatical partial trace leads to a straight line, verified by

$$tr_A(|\Omega\rangle_{CA\ CA}\langle\Omega|) = \frac{1}{d}\sum_{i,j=0}^{d-1} tr_A(|e_i^C \otimes e_i^A\rangle\langle e_j^C \otimes e_j^A|) = \frac{1}{d}(\mathbb{1}_d)_C, \tag{12}$$

and the second type of diagrammatical partial traces yield oblique lines for the transfer operators T_{CB} and T_{BC}, which are algebraically represented by

$$\frac{1}{d}T_{CB} = tr_A(|\Omega\rangle_{CA\ AB}\langle\Omega|), \qquad \frac{1}{d}T_{BC} = tr_A(_{CA}\langle\Omega|\Omega\rangle_{AB}). \tag{13}$$

See Figure 7 for how to form closed circles in three distinct ways. A top cup with a bottom cap forms the same closed circle as a top cap and a bottom cup, as is revealed in the algebraic expression,

$$tr_{CA}((\rho_C \otimes \mathbb{1}_d|\Omega\rangle_{CA})(_{CA}\langle\Omega|\mathbb{1}_d \otimes \mathcal{O}_A^T)) = {}_{CA}\langle\Omega|(\mathbb{1}_d \otimes \mathcal{O}_A^T)(\rho_C \otimes \mathbb{1}_d)|\Omega\rangle_{CA}, \tag{14}$$

where ρ_C and $\mathcal{O}_A$ are bounded linear operators in the d-dimensional Hilbert space. A closed circle formed by two oblique lines denotes the same trace as a top cap with a bottom cup, as can be algebraically proved,

$$tr_{CA}((\rho_C T_{CA})(\mathcal{O}_A T_{AC})) = d \cdot {}_{CA}\langle\Omega|(\rho_C \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathcal{O}_A^T)|\Omega\rangle_{CA}. \tag{15}$$

In the extended TL diagrammatical representation for a quantum circuit, we perform its presumed functions using topological-like operations defined as above.
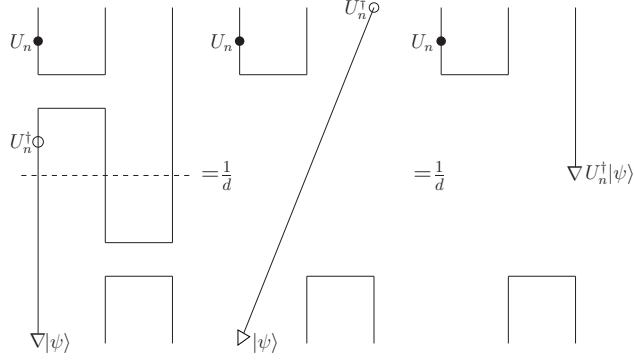
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Figure 8: "Topological" description for the teleportation.

4 "Topological" descriptions for quantum circuits

We study topological-like descriptions for the teleportation and entanglement swapping which are stimulating examples for extended TL diagrammatical quantum circuits.

4.1 "Topological" description for teleportation

Teleportation [15] can be observed from the viewpoint of quantum measurement [16, 17]. The maximally entangled state $|\Omega\rangle_{AB}$ shared by Alice and Bob is created in the quantum measurement denoted by the projector $(|\Omega\rangle\langle\Omega|)_{AB}$, while the quantum measurement performed by Alice in the composite system of Charlie and herself is represented by the projector $(|\Omega_n\rangle\langle\Omega_n|)_{CA}$. Therefore the teleportation equation has the following formulation,

$$(|\Omega_n\rangle\langle\Omega_n| \otimes \mathbb{1}_d)(|\psi\rangle \otimes |\Omega\rangle\langle\Omega|) = \frac{1}{d}(|\Omega_n\rangle \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes (\mathbb{1}_d \otimes U_n^\dagger|\psi\rangle)\langle\Omega|), \qquad (16)$$

where the lower indices A, B, C are omitted for convenience and there are d^2 distinguished classical channels between Alice and Bob due to $n = 1, \cdots, d^2$.

We make a "topological" description for the teleportation based on quantum measurements. It is encoded in Figure 8 where the symbols Ω and T are omitted for simplicity, by reading the teleportation equation (16) from the left hand-side to the right hand-side and drawing Figure 8 from the top to the bottom in view of extended TL diagrammatical rules. Move the unitary operator $U_n^\dagger$ from Charlie's system to Bob's system along the path formed by a top cap and a bottom cup; apply the "topological" operation by straightening the configuration of the top cap and bottom cup into an oblique line; transport a unknown quantum state $|\psi\rangle_C$ along the oblique line from Charlie to Bob. Finally, Charlie has a quantum state $U_n^\dagger|\psi\rangle_B$ and applies the local unitary transformation U_n to obtain $|\psi\rangle_B$. Note that in the extended TL diagrammatical recipe for the teleportation, the cup and the cap do not straighten to the identity but rather to a unitary transformation that depends upon the measurement outcome.

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Furthermore, Figure 8 is presenting a suitable diagrammatical framework for unifying various sorts of descriptions for teleportation, see [8, 9] for the detail. It can include Vaidman's continuous teleportation [16, 17] and other discrete teleportation schemes. Its enclosure (Figure 10) represents Werner's tight teleportation scheme [24]. Removing "irrelevant" parts (the top cup with the solid point for U_n and the bottom cap) leads to the well known configuration for the quantum information flow in the literature [2, 3, 4, 5, 6, 14, 7].

Moreover, a cup-over-cap represents a projector which formally denotes a quantum measurement and has little to do with how states are actually prepared and measured in the laboratory. It is a generator of the TL algebra and hence Figure 8 is typical configuration of the extended TL algebra, i.e., the TL algebra under local unitary transformations, see Appendix A. Hence a natural connection between quantum information and the TL algebra has been recognized this way [8, 9].

4.2 "Topological" description for entanglement swapping

Entanglement swapping [19] produces the entanglement between two independent systems as a consequence of quantum measurements instead of physical interactions. Alice has a maximally entangled bipartite state $|\Omega_l\rangle_{ab}^A$ for particles a, b and Bob has $|\Omega_m\rangle_{cd}^B$ for particles c, d. They are independently created and do not share common history. Alice applies a quantum measurement denoted by $\mathbb{1}_d \otimes (|\Omega_n\rangle\langle\Omega_n|)_{bc} \otimes \mathbb{1}_d$ to the product state of $|\Omega_l\rangle_{ab}^A$ and $|\Omega_m\rangle_{cd}^B$ so that the entanglement swapped state $|\Omega_{lnm}\rangle_{ad}^{AB}$ is a maximally entangled bipartite state shared by Alice and Bob for particles a, d, i.e.,

$$
(\mathbb{1}_d \otimes (|\Omega_n\rangle\langle\Omega_n|)_{bc} \otimes \mathbb{1}_d)(|\Omega_l\rangle_{ab}^A \otimes |\Omega_m\rangle_{cd}^B)
$$
$$
= \frac{1}{d}(\mathbb{1}_d \otimes |\Omega_n\rangle_{bc} \otimes \mathbb{1}_d)\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}(U_l U_n^* U_m |e_i\rangle_a^A \otimes \mathbb{1}_d \otimes \mathbb{1}_d \otimes |e_i\rangle_d^B)
$$
$$
\equiv \frac{1}{d}(\mathbb{1}_d \otimes |\Omega_n\rangle_{bc} \otimes \mathbb{1}_d) \, |\Omega_{lnm}\rangle_{ad}^{AB}. \tag{17}
$$

In other words, the entanglement swapping reduces a four-particle state $|\Omega_l\rangle_{ab}^A \otimes |\Omega_m\rangle_{cd}^B$ to a bipartite entangled state $|\Omega_{lnm}\rangle_{ad}^{AB}$ using the entangling quantum measurement.

In the extended TL diagrammatical representation for the entanglement swapping, Figure 9 in which the omission of symbols Ω has no confusion, the entanglement swapping equation (17) can be proved at the diagrammatical level by collecting unitary operators U_l, U_n^* and U_m at the system for the particle a, and then applying "topological" diagrammatical operations in Figure 6. Note that Figure 9 is a standard Temperely–Lieb configuration under local unitary transformations and the entanglement swapping presents a typical example for the quantum network consisting of maximally entangled states and local unitary transformations, see Appendix A or [8, 9].
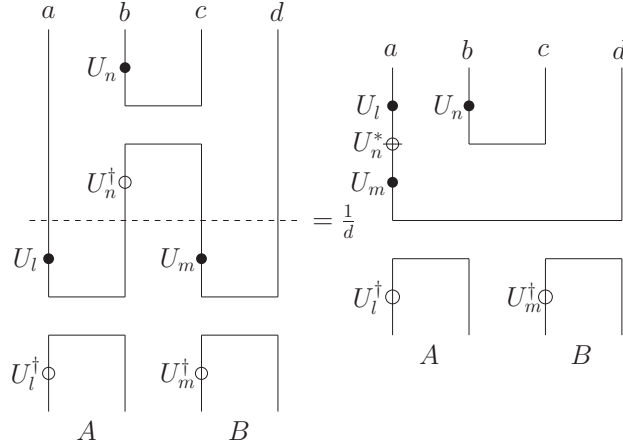
Figure 9: "Topological" description for the entanglement swapping.

5 Characteristic equations for quantum circuits

A characteristic equation for a quantum circuit is derived by applying a series of topological-like operations to the enclosure of its extended TL diagrammatical representation. This equation includes all essential elements for this quantum circuit. We study characteristic equations for the teleportation, dense coding and entanglement swapping.

Here all involved finite Hilbert spaces are d dimensional and the classical channel distinguishes d^2 signals, which are called the tight scheme for quantum information protocols by Werner [24]. The density operator ρ has a form $\rho = |\phi_1\rangle\langle\phi_2|$. The channel T_n describes a local unitary transformation U_n on an observable $\mathcal{O}$, which are given by

$$T_n(\mathcal{O}) = U_n^\dagger \mathcal{O} U_n, \qquad \mathcal{O} = |\psi_1\rangle\langle\psi_2|, \qquad n = 1, 2, \cdots d^2. \tag{18}$$

5.1 Characteristic equation for teleportation

In the tight teleportation scheme [24], Charlie has his density operator $\rho_C = (|\phi_1\rangle\langle\phi_2|)_C$ to include a quantum state sent to Bob, while Alice and Bob share the maximally entangled state $\omega_{AB} = (|\Omega\rangle\langle\Omega|)_{AB}$. Alice chooses her observables $(\omega_n)_{CA}$ to make the Bell measurement in the composite system between Charlie and her and then passes the message labeled by n on to Bob via the classical channel. Finally, Bob performs a unitary correction on his observable $\mathcal{O}_B$ by the quantum channel T_n. In terms of ρ_C, ω_{AB}, $(\omega_n)_{CA}$ and $T_n(\mathcal{O}_B)$, the tight teleportation scheme is summarized in the characteristic equation,

$$\sum_{n=1}^{d^2} tr((\rho_C \otimes \omega_{AB})((\omega_n)_{CA} \otimes T_n(\mathcal{O}_B))) = tr(\rho_C \mathcal{O}_B), \tag{19}$$

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$$tr((\rho \otimes \omega)(\omega_n \otimes T_n(\mathcal{O}))) = \frac{1}{d^2} tr(\rho \mathcal{O})$$

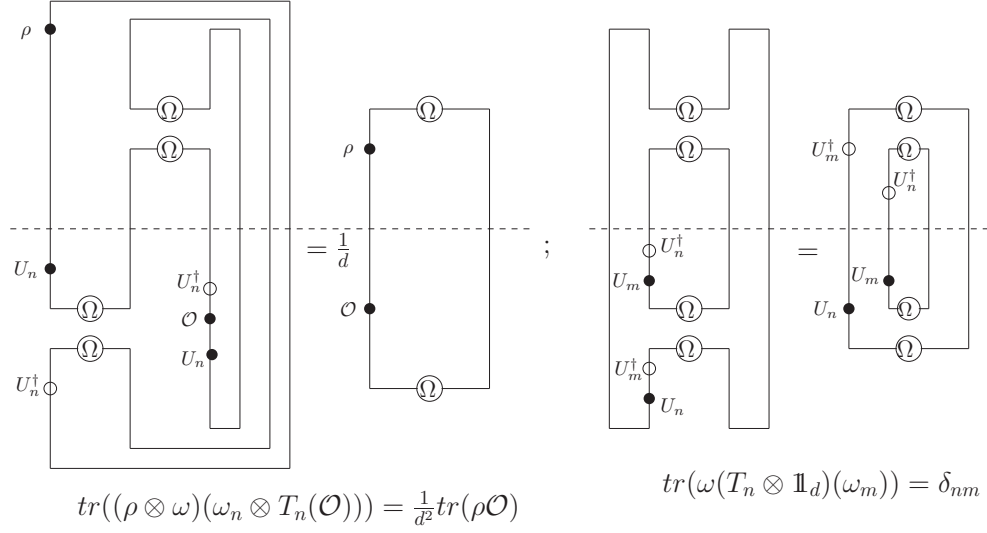$$tr(\omega(T_n \otimes \mathbb{1}_d)(\omega_m)) = \delta_{nm}$$

Figure 10: Characteristic equations for teleportation and dense coding.

which catches the crucial point of a successful teleportation, i.e., Charlie makes the measurement in his system as he does in Bob's system although they are independent from each other.

See the left term of Figure 10 where the lower indices A, B, C are omitted for convenience. The enclosure of its extended TL diagrammatical representation Figure 8, is obtained by connecting top boundary points to bottom boundary points in the systems for Charlie, Alice and Bob, respectively. Working on such the enclosure, we have two diagrammatical approaches of deriving the characteristic equation (19) for the teleportation. The first way is to move the local unitary operators $U_n^\dagger$ and U_n along the configuration of cups or caps until they meet to yield the identity, apply "topological" operations suggested by the second or third term of Figure 6 and then exploit the right term of Figure 7. The second way is to combine "topological" diagrammatical operations suggested by the first term of Figure 6 with the left term of Figure 7. In addition, we arrange the density operator ρ and observable $\mathcal{O}$ in the same straight line by moving them along branches of cups or caps.

5.2 Characteristic equation for dense coding

The tight dense coding [24] can be also performed using topological-like operations. Alice and Bob share the maximally entangled state $|\Omega\rangle_{AB}$, and Alice transforms her state by the channel T_n to encode a message n and then Bob makes the quantum measurement on an observable ω_m of his system. At $n = m$, Bob gets the message. See the right term of Figure 10: The process of this kind of dense coding is concluded in its diagrammatical and algebraic characteristic equations, with "topological" operation by the left term of Figure 7 to be exploited.

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$$tr((\rho \otimes \omega_n \otimes T_n(\mathcal{O}))(\omega \otimes \omega)) = \tfrac{1}{d^3} tr(\rho \mathcal{O}^T)$$
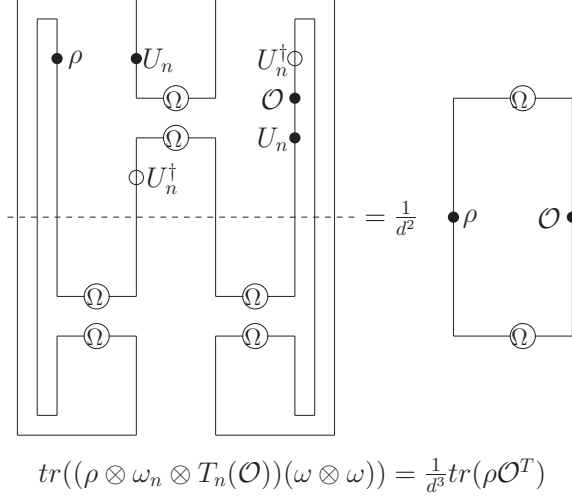
Figure 11: Characteristic equation for the entanglement swapping.

5.3 Characteristic equation for entanglement swapping

See Figure 11 in which the lower indices a, b, c, d are neglected and the transpose $\mathcal{O}^T$ of the observable $\mathcal{O}$ is defined by

$$\mathcal{O}^T = \sum_{i,j=0}^{d-1} \psi_{1i}\psi_{2j}^* |e_j\rangle\langle e_i|, \qquad (\langle e_i|)^T \equiv |e_i\rangle. \tag{20}$$

It is the enclosure of its extended TL diagrammatical representation Figure 9. Exploit "topological" operations by Figure 5 and then that by the left term of Figure 7 to derive its characteristic equation,

$$\sum_{n=1}^{d^2} tr((\rho_a \otimes (\omega_n)_{bc} \otimes T_n(\mathcal{O}_d))(\omega_{ab} \otimes \omega_{cd})) = \frac{1}{d} tr(\rho_a \mathcal{O}_d^T), \tag{21}$$

where the summation is over n^2 classical channels. Note that the entanglement swapping can be used to detect the Bell inequality [1] although entanglement is yielded via quantum measurements.

6 Comments on known diagrammatical approaches

This section is aimed at commenting on several known diagrammatical approaches devised for describing quantum information phenomena in the recent literature by pointing out essential differences and connections among them. All of them are believed to be various generalizations of the diagrammatical technique in relation to tensor categories well-known in the mathematical literature, stemming from the work by Joyal

and Street [25] in the 1990's, with important contributions by Turaev and others, (and also going back to pioneering work by Kelly in the 1970's [26]), see Kassel' textbook [27] and Turaev's book [28] for more relevant references.

Category is a sort of the abstract language to describe a collection of mathematical objects as well as structure-preserving morphisms between them. Definitions of various kinds of categories used in the following have been sketched in Appendix B, they including categories, monoidal categories (tensor categories), pivotal categories, dagger pivotal categories, braided monoidal categories, symmetric monoidal categories, symmetric pivotal categories (compact closed categories), symmetric dagger pivotal categories (strongly compact closed categories or 3-tuply categories with duals) and d-dimensional cobordism categories.

6.1 Categorical quantum mechanics & information

Categorical structures for quantum information phenomena can be set up by regarding physical systems (such as qubits) as objects and physical operations (such as local unitary transformations) as morphisms. To propose high-level methods for quantum computation and information, Abramsky and Coecke [2, 3, 4] refine strongly compact categories and exploit them to comprehensively axiomatize quantum mechanics and study quantum information protocols, and also make the detailed elaboration of diagrammatical representation for tensor categories to quantum mechanics & information.

Categorical quantum mechanics is a typical example for strongly compact closed categories, and it has all Hilbert spaces as objects and linear bounded operators as morphisms. It is equipped with dagger and dual operations to capture the complex structure of quantum mechanics where one has transpose and complex conjugation as separate things with the adjoint distinct from the dual. The diagrammatical representation for categorical quantum mechanics can be viewed as a two-dimensional generalization of the Dirac notation.

Strongly compact closed categories are also called 3-tuply categories with duals by Baez and Dolan [29, 30] or dagger compact closed categories by Selinger [31].

6.2 Atemporal diagrammatical approach

Griffiths et al. [7] devise a system of atemporal diagrams to describe various elements of a quantum circuit where "atemporal" means such a diagrammatical representation makes no reference to time. This kind of diagrammatical representation is also a sort of generalization of the diagrammatical recipe for tensor categories. But it allows oriented diagrams with an arrow from a Hilbert space to its adjoint space, and explicitly reveals the map-state duality in which a maximally bipartite entangled state can be identified with a unitary map. Especially, its diagrammatical prescription on completely positive map, positive operator, super operator, transition operator and dynamical operator is almost equivalent to Selinger's CPM construction [31] over strongly compact closed categories.

In the atemporal diagrammatical approach [7], the maximally entangled state plays

the role of the transposer denoted by $|\mathcal{A}\rangle$, a quantum (classical) channel is simply denoted by lines, and any type of operators can be represented since lines are allowed to point to any directions. In the extended TL diagrammatical representation [8, 9], however, the maximally entangled state $|\Omega\rangle$ (or $\langle\Omega|$) plays the kernel role with the cup (or cap) configuration [4], the quantum channel is a TL configuration with solid points (or small circles), and labels for distinct Hilbert spaces are arranged at a horizontal line so that "topological" features in diagrammatical quantum circuits can be explicitly observed.

Moreover. Removing "irrelevant parts" from Figure 8 leads to the same quantum channel as Fig. 7(d) [7] if one represents Ψ by a cup and denote Φ_j by a cap with a small circle for the local unitary transformation $U_j^\dagger$, and as $j = k$ the teleportation is performed. As one makes a double of Fig. 7(d) [7] with an additional density operator and observable, he will obtain Werner's tight teleportation scheme [24], i.e., the left term of Figure 10.

6.3 The extended Temperley–Lieb categorical approach

In this paper together with [8, 9], motivated by seeking for topological and algebraic structures underlying various quantum information phenomena and then obtaining helpful insights for the application of unitary Yang–Baxter solutions as universal quantum gates to quantum information and computing, extended TL diagrammatical rules are set up for quantum information protocols in terms of maximally entangled states and local unitary transformations. Topological-like descriptions are made for the teleportation, dense coding and entanglement swapping using topological-like operations and characteristic equations are derived for them.

In [8, 9], the TL algebra with physical operations (such as local unitary transformation) is found to present a suitable mathematical framework for quantum information protocols including quantum teleportation and entanglement swapping. The TL category denotes a collection of all TL algebras: the unoriented TL category is a free dagger pivotal category over self-dual object, i.e., the oriented TL category is a free dagger pivotal category. Therefore, the extended TL category represents a collection of all TL algebras with physical operations. Furthermore, the teleportation configuration [8, 9] can be regarded as the defining configuration for the diagrammatical representation of the Brauer algebra [13] which is the extension of the TL algebra with a symmetry (permutation or swap or flat crossing) generator. Similarly, the Brauer category denoting the collection of all Brauer algebras, has oriented and unoriented generalizations. Additionally, the Brauer category is a kind of the extended TL category because a symmetry (swap) can be represented by a linear combination of the extended TL configurations (see [9] for an example).

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[4]The maximally entangled state $|\Omega\rangle$ and its adjoint state $\langle\Omega|$ can be respectively regarded as the unit and counit mappings defining the pivotal category (see Appendix B) in which the dual of a morphism is understood to be its ordinary transposition. This is an important connection between the atemporal diagrammatical approach [7] and the extended TL categorical approach [8, 9].

Although cup (cap) configurations are exploited in diagrammatical representations for both categorical quantum mechanics and extended TL categorical approach, strongly compact closed categories are symmetric but the TL categories are planar without symmetry. More significantly, in the description of quantum information protocols [8, 9] like quantum teleportation and entanglement swapping, a symmetry operator is not needed and the transposition M^T of an operator M is found to play the key role[5]. Moreover, essential differences between two approaches to the description of quantum information flow have been presented in detail, see [8, 9], where the quantum information flow is created together with loops, "irrelevant" cups and caps, and the normalization factor allowed to be zero.

Note that conceptual differences between Kauffman and Lomonaco's teleportation topology [5] and the extended TL categorical approach are explained clear [8, 9], together with both approaches appreciating the same matrix diagrammatical technique.

6.4   Cob[0]: 0-dimensional cobordism category

Kauffman and Lomonaco [6] remark on the 0-dimensional cobordism category **Cob[0]** with a natural relationship to quantum mechanics and quantum teleportation, i.e., directly related to the Dirac notation of bras and kets. The one object of the **Cob[0]** category is a single point p, i.e., the simplest zero dimensional manifold, and the other object is the empty set $*$ (the empty manifold). A morphism between the point p and another point q is the line segment with boundary points p and q, the identity morphism to be a map from p to p. It is simple to identity various morphisms between points and the empty set $*$ with Dirac bras and kets, together with the scalar product recognized to be a morphism between $*$ and $*$. Note that the **Cob[0]** category is the Brauer category, in other words, the **Cob[0]** category without crossings between any of line segments is the TL category, furthermore, it is also a sort of the extended TL category.

6.5   Summarizing comments from the perspective of TQFT

As a summary of our comments on known diagrammatical approaches, we know from the first footnote that what is relating categorical quantum mechanics to the extended TL categorical approach (i.e., the **Cob[0]** category or Brauer category) is a well known functor used to define TQFT[6] by Atiyah [20]. He defines a d-dimensional TQFT as a functor from the d-dimensional cobordism category **Cob[d]** to the category **Vect** of all vector spaces and linear mappings, i.e., a representation of the **Cob[d]**.

Furthermore, a d-dimensional TQFT is also a functor from the **Cob[d]** category to the category **Hilb** of all Hilbert spaces and linear bounded operators, and this functor

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[5]This is consistent with the fact that the TL category is a pivotal category in which no symmetry is imposed and the dual of a morphism can be identified with its transposition.

[6]Also, once we acknowledge that the functor defining TQFT plays the key role in quantum physics, we may be suggesting the nature of quantum physics to be determined by the topology of the ground state of a given physical system, for example that quantum orders describing distinct phases in the ground state of many-body systems may be recognized to be topological orders [32].

can be equipped with two types of dualities which are exploited to define the transpose, complex conjugation, and adjoint operation for all morphisms. Here we call this sort of functor by the strongly compact closed functor[7] because both the **Cob[d]** category and **Hilb** category are strongly compact closed categories.

Moreover, it is worthwhile noting [8] that, state preparation and quantum measurements are essential parts of quantum information and computation, but which can not be directly described by categorical structures which operate with objects, and morphisms and functors. This means that in categorical quantum mechanics we have two conceptual levels: the one is devised for categorical descriptions; the other is specially designed for state vectors in the Hilbert space. In other words, in the known categorical definition of TQFT, state preparation and quantum measurement have been not considered as seriously as in categorical quantum mechanics & information.

7   Categorical quantum physics & information

In the last section, first of all, ambiguities have to be clarified for why the proposal in the first footnote chooses the name *categorical quantum physics and information* instead of categorical quantum mechanics & information suggested by Abramsky and Coecke [2, 3, 4]. As the unit object of pivotal categories (see Appendix B) is identified with the vacuum, the unit morphism η_A (the counit morphism ϵ_A) can be explained as the creation (annihilation) of a pair of a particle and its anti-particle from (into) the vacuum with the input (output) of enough energy, which is exploiting the language of quantum field theory instead of quantum mechanics. Besides this, another explicit point is that these categorical structures (as above or see Appendix B) are not just important in the formulation of basic quantum mechanics itself (oriented towards quantum information) but also for exotic constructions towards TQFT, string theories and quantum gravity.

As we see in this paper, categorical structures have been intensively exploited in the study of quantum information and computation. Strongly compact closed categories have an example of the category **Hilb** including all Hilbert spaces and linear bounded operators for recasting axioms of quantum mechanics in the abstract language. The extended TL categories are found to be especially fitted for quantum information protocols like quantum teleportation and entanglement swapping. The "partially" braided monoidal categories [9] is suggested to be a mathematical framework for describing quantum circuits consisting of single qubit transformations and non-trivial unitary braids as universal two-qubit quantum gates. In addition, modular tensor categories (see

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[7]According to the proposal in the first footnote about *dagger ribbon categories for categorical quantum physics and information*, this functor would be better called the *dagger ribbon functor*.

[8]This note is suggesting there still remain many fundamental conceptual problems to be solved and clarified in categorical quantum mechanics (physics), for example, how to fix the global phase in the superposition principle of state vectors in a Hilbert space, see [3].

[9]They are based on [5, 33, 34] which study universal quantum computation by combining non-trivial unitary braids as two-qubit universal quantum gates with single qubit transformations. "Partially" means that the naturality condition for defining the braided monoidal category can not be satisfied because a tensor product of single qubit transformations does not often commute with a two-qubit braiding gate.

Appendix B) for 2-dimensional TQFT (for example, the $SU(2)$ Witten-Chern-Simons theory at roots of unity [35]) are the mathematics framework for topological quantum computing by Freedman, Larsen and Wang [36, 37].

Besides these applications of various category theories to quantum information phenomena, conformal field theories defined by Segal [38] describe processes in string theories as morphisms; loop gravity [39] has spin network as objects and spin foams as morphisms; a higher-dimensional categorical notion called n-category is devised by Baez and Dolan [29, 30] for reconciling the general relativity and quantum mechanics. Especially, a kind of physical interpretation for every element of tensor categories has been proposed by Levin and Wen [32] to explain topological orders in condensed matter physics via string-net condensation.

Therefore, we think that we are able to make acceptable reasons for the proposal in the first footnote that *dagger ribbon categories* (i.e., strongly ribbon categories or ribbon categories with the dagger operation, see Appendix B) are a suitable mathematical framework to describe *categorical quantum physics and information* [10]. The combination of the left duality with the dagger operation ensures the complex structure and complex conjugation crucial for quantum physics. It is well known, strongly compact closed categories for categorical quantum mechanics & information [2, 3, 4] are special examples of *dagger ribbon categories* where the braiding is a symmetry (permutation) and the twist operator is the identity, and modular tensor categories with the dagger operation, responsible for the mathematical description of topological quantum computing [36, 37], are also special cases for *dagger ribbon categories*.

But this proposal will raise a natural question about the roles that the braiding and twist play in *categorical quantum physics and information*. Possible answers have been discussed in the first footnote. "Braiding" suggests that physical objects either fundamental ones at the Planck energy scale or quasi-particles of many-body systems are required to obey the braiding statistics, while 'twist" means that they are either string-like (even braine-like), i.e., extended configurations instead of point particles, or have an effective (a new internal) degree of freedom called the "twist spin". The latter one can be commented from the similar historical story how an electron was found to have a spin quantum number different from its known orbital angular momentum quantum number. Therefore, a quasi-particle obeying the braiding statistics like an anyon either has the "twist spin", or behaves in a string-like way so that the configuration formed by its motion can be denoted by a strip or ribbon in which the existence of the "twist spin" is natural, in addition, this anyon should live with the so called quantum dimension specified by ribbon tensor categories.

As concluding remarks, *categorical quantum physics* has been explained word by word: "categorical" by *dagger ribbon categories* [11]; "quantum" by the superposition

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[10]In categorical quantum physics and information, it is necessary to introduce state vectors of the Hilbert space (which is only an object of the dagger ribbon category) to define state preparation and quantum measurement. How to deal with classical communication is also an interesting problem to be discussed in the categorical language or quantum approach.

[11]The classification of dagger ribbon categories will be a worthwhile problem to be considered for both mathematicians and physicists.

principle of state vectors in a Hilbert space as well as the complex structure (such as the imaginary unit i) and unitary evolution of a state vector; and "physics" by specific physical topics to be described in the framework of categories. Furthermore, it is explicit that the name *categorical quantum physics and information* survives different sorts of interpretations, for example, "categorical" can be related to other categorical structures (even categories over categories) in which *dagger ribbon categories* are subcategories or special examples so that quantum measurement, classical data, etc., can be treated at the same time. Moreover, it is crucial to emphasize again that the proposal for *categorical quantum physics and information* in the first footnote is motivated by the present study in quantum information phenomena and theory, to be a solution for the problem how to coordinate a mathematical framework in which both categorical quantum mechanics [2, 3, 4] and topological quantum computing [36, 37] are interesting examples. Finally, dagger ribbon categories involved in this article are those with positive definite forms between two morphisms, i.e., positive dagger ribbon categories which are called unitary Hermitian ribbon categories in Turaev's book [28].

Acknowledgements

A   The extended TL algebra

The extended TL algebra, i.e., the TL algebra with local unitary transformations, has been proposed to underlie quantum circuits in terms of maximally entangled states and local unitary transformations [8, 9]. The TL algebra TL_n is generated by identity Id and $n-1$ hermitian projectors E_i satisfying

$$
\begin{aligned}
E_i^2 &= E_i, & (E_i)^\dagger &= E_i, \ \ i = 1, \ldots, n-1, \\
E_i E_{i\pm 1} E_i &= \lambda^{-2} E_i, & E_i E_j &= E_j E_i, \ \ |i - j| > 1,
\end{aligned}
\tag{22}
$$

in which λ is called the loop parameter.

A representation of the $TL_n(d)$ algebra is obtained in terms of the maximally entangled state ω, a projector, by defining idempotents E_i in the way

$$
E_i = (Id)^{\otimes(i-1)} \otimes \omega \otimes (Id)^{\otimes(n-i-1)}, \qquad i = 1, \cdots n-1.
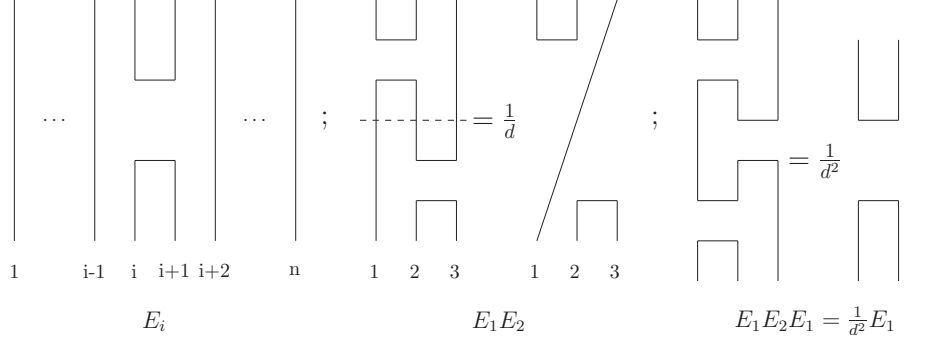\tag{23}
$$

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Figure 12: Teleportation configuration and TL algebra.

For example, the $TL_3(d)$ algebra is generated by two idempotents E_1 and E_2,

$$E_1 = \omega \otimes Id, \qquad E_2 = Id \otimes \omega. \tag{24}$$

In Figure 12, there are diagrammatical representations for E_i, $E_1 E_2$ and $E_1 E_2 E_1 = \frac{1}{d^2} E_1$ with the loop parameter d. $E_1 E_2$ has a normalization factor $\frac{1}{d}$ from a vanishing cup and a vanishing cap, and $E_1 E_2 E_1$ has a factor $\frac{1}{d^2}$ from two vanishing cups and two vanishing caps.

With local unitary transformations U_n of the maximally entangled state ω, one can set up another representation of the TL algebra. For example, the $TL_3(d)$ algebra is generated by $\tilde{E}_1$ and $\tilde{E}_2$,

$$\tilde{E}_1 = \omega_n \otimes Id, \qquad \tilde{E}_2 = Id \otimes \omega_n. \tag{25}$$

Therefore, the extended TL algebra has diagrammatical configurations consisting of cups, caps and solid points or small circles which have been exploited to describe quantum circuits [8, 9].

Furthermore, the extended TL algebra is also an interesting mathematical framework for performing quantum computing at the diagrammatical level. For example, the CNOT gate, a linear combination of tensor products of Pauli matrices σ_1, σ_2 and σ_3,

$$C = \frac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_1 + \sigma_3 \otimes \mathbb{1}_2 - \sigma_3 \otimes \sigma_1) \tag{26}$$

which satisfies the properties of the CNOT gate,

$$C|00\rangle = |00\rangle, \; C|01\rangle = |01\rangle, \; C|10\rangle = |11\rangle, \; C|11\rangle = |10\rangle, \tag{27}$$

has an extended TL diagrammatical representation Figure 13. Similarly, a swap gate (symmetry) can be represented by a linear combination of the extended TL configuration, see [9] for an example, which verifies that the Brauer category [13] is a kind of extended TL category.

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$$C = \tfrac{1}{2}(\mathbb{1}_2 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_1 + \sigma_3 \otimes \mathbb{1}_2 - \sigma_3 \otimes \sigma_1)$$

Figure 13: The CNOT gate in our diagrammatical approach.

B   Definitions of Categories

Definitions of various kinds of categories used in this paper have been sketched in the following. They include categories, monoidal categories (tensor categories), pivotal categories, dagger pivotal categories, braided monoidal categories, symmetric monoidal categories, symmetric pivotal categories (compact closed categories), symmetric dagger pivotal categories (strongly compact closed categories or 3-tuply categories with duals), ribbon categories, modular tensor categories and d-dimensional cobordism categories. References for them are referred to Kassel's textbook [27] and Turaev's book [28].

A *category* $\mathcal{C}$ consists of objects $A, B, C, \cdots$, associative morphisms $f, g, h, \cdots$,

$$\frac{f : A \to B, \ g : B \to C,}{g \circ f : A \to C,}, \qquad (h \circ g) \circ f = h \circ (g \circ f), \qquad (28)$$

and identity morphism Id between these objects,

$$f \circ Id_A = f, \qquad Id_B \circ f = f, \qquad Id_A : A \to A, \qquad Id_B : B \to B. \qquad (29)$$

A morphism is also called an arrow. The inverse of a morphism f is denoted by $f^{-1} : B \to A$. A *functor* Z between two categories $\mathcal{C}$ and $\mathcal{C}'$ is a structure-preserving map in the following sense,

$$\frac{f : A \to B}{Z(f) : Z(A) \to Z(B)}, \qquad Z(f \circ g) = Z(f) \circ Z(g), \ Z(Id_A) = Id_{Z(A)} \qquad (30)$$

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, l, r)$ is a category $\mathcal{C}$ equipped with a tensor product $\otimes$ on objects and morphisms,

$$\frac{f : A \to B, \qquad g : C \to D}{f \otimes g : A \otimes C \to B \otimes D} \qquad (31)$$

together with unit object I, associative natural isomorphism α, left and right natural unit isomorphisms l and r,

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C), \qquad l_A : A \xrightarrow{\cong} I \otimes A, \qquad r_A : A \xrightarrow{\cong} A \otimes I, \quad (32)$$

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where the symbol $\cong$ denotes the isomorphism relation. The associator α has to satisfy the pentagon equation on the tensor product $((A \otimes B) \otimes C) \otimes D$,

$$(Id_A \otimes \alpha_{B,C,D}) \circ (\alpha_{A,B\otimes C,D}) \circ (\alpha_{A,B,C} \otimes Id_D) = (\alpha_{A,B,C\otimes D}) \circ (\alpha_{A\otimes B,C,D}), \quad (33)$$

while the associator α, left and right units l, r have to satisfy the triangle equation on the tensor product $A \otimes I \otimes B$,

$$(Id_A \otimes l_B) \circ (\alpha_{A,I,B}) = (r_A \otimes Id_B). \quad (34)$$

The pentagon and triangle equations are also called coherent laws. As the associator α is an identity morphism, the monoidal category is called strict. In the literature, monoidal categories are also called *tensor categories*.

A *pivotal category* is a monoidal category with the left duality $()^*$ which assigns a dual object A^* to any object A and imposes a unit morphism η_A and a counit morphism ϵ_A given by

$$\epsilon_A : A^* \otimes A \to I, \qquad \eta_A : I \to A \otimes A^* \quad (35)$$

satisfying the triangular identities,

$$\begin{aligned} l_{A^*}^{-1} \circ (\epsilon_A \otimes Id_{A^*}) \circ \alpha_{A^*,A,A^*}^{-1} \circ (Id_{A^*} \otimes \eta_A) \circ r_{A^*} &= Id_{A^*}, \\ r_A^{-1} \circ (Id_A \otimes \epsilon_A) \circ \alpha_{A,A^*,A} \circ (\eta_A \otimes Id_A) \circ l_A &= Id_A, \end{aligned} \quad (36)$$

which are also called the rigid conditions. The left duality $()^*$ acts on objects in the manner,

$$A^{**} \cong A, \qquad (A \otimes B)^* = B^* \otimes A^*, \qquad I^* = I \quad (37)$$

and it is a contravariant involutive functor,

$$(Id_A)^* = Id_{A^*}, \qquad (g \circ f)^* = f^* \circ g^*, \qquad f^{**} = f \quad (38)$$

where the transpose $f^* : B^* \to A^*$ of the morphism $f : A \to B$ is defined by

$$f^* = (\epsilon_B \otimes Id_{A^*}) \circ (Id_{B^*} \otimes f \otimes Id_{A^*}) \circ (Id_{B^*} \otimes \eta_A). \quad (39)$$

A category can be equipped with an involutive, identity-on-objects, contravariant functor $()^\dagger : \mathcal{C} \to \mathcal{C}^{op}$ which defines the adjoint $f^\dagger : B \to A$ of a morphism $f : A \to B$ satisfying

$$A^\dagger \cong A, \qquad (Id_A)^\dagger = Id_A, \qquad (g \circ f)^\dagger = f^\dagger \circ g^\dagger, \qquad f^{\dagger\dagger} = f. \quad (40)$$

This adjoint functor $()^\dagger$ can coherently preserve the tensor product structure in a monoidal category,

$$(A \otimes B)^\dagger \cong A \otimes B, \qquad (f \otimes g)^\dagger \cong f^\dagger \otimes g^\dagger, \quad (41)$$

and the adjoints of the associator α, left and right units l, r are given by

$$\alpha^\dagger = \alpha^{-1}, \qquad l^\dagger = l^{-1}, \qquad r^\dagger = r^{-1}. \quad (42)$$

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A *dagger pivotal category* is called for a monoidal category with the left duality ()* and ()† functor, which allows $\eta_A^\dagger = \epsilon_{A^*}$ and $\epsilon_A^\dagger = \eta_{A^*}$ and leads to a covariant functor ()$_*$ denoting the complex conjugation f_* of the morphism f,

$$\frac{f : A \to B}{f_* : A^* \to B^*}, \qquad f \mapsto f^{*\dagger}; \qquad (f^*)_* = f^\dagger = (f_*)^*. \tag{43}$$

A *braided monoidal category* is a monoidal category with a braiding, $\sigma_{A,B} : A \otimes B \to B \otimes A$, a natural isomorphism satisfying

$$(g \otimes f) \circ \sigma_{A,B} = \sigma_{B',A'} \circ (f \otimes g), \qquad f : A \to B', \ g : B \to A' \tag{44}$$

for all morphisms f and g, and also satisfying the hexagonal equations,

$$\begin{aligned}
(Id_B \otimes \sigma_{A,C}) \circ \alpha_{B,A,C} \circ (\sigma_{A,B} \otimes Id_C) &= \alpha_{B,C,A} \circ \sigma_{A,B \otimes C} \circ \alpha_{A,B,C}, \\
(\sigma_{A,C} \otimes Id_B) \circ \alpha_{A,C,B}^{-1} \circ (Id_A \otimes \sigma_{B,C}) &= \alpha_{C,A,B}^{-1} \circ \sigma_{A \otimes B,C} \circ \alpha_{A,B,C}^{-1}.
\end{aligned} \tag{45}$$

As the braiding $\sigma_{A,B}$ forms a representation of the symmetrical group, i.e., satisfying $\sigma_{A,B} \circ \sigma_{B,A} = Id_{A \otimes B}$ for all objects A and B, this category is called the *symmetric monoidal category*.

In the literature, symmetric pivotal categories are called *compact closed categories* or *rigid symmetric monoidal categories*, and *dagger symmetric pivotal categories* [31] are called *strongly compact closed categories* by Abramsky and Coecke [2, 4] or *3-tuply categories with duals* by Baez and Dolan [29]. The *compact closed category* was introduced by Kelly [26] in the 1970's, in particular an important paper by Kelley and Laplaza on coherence in compact closed categories [40].

A twist θ_A in the braided monoidal categories is a natural isomorphism $\theta_A : A \to A$ satisfying

$$\theta_{A \otimes B} = (\theta_A \otimes \theta_B) \circ \sigma_{B,A} \circ \sigma_{A,B}, \qquad \theta_{A^*} = (\theta_A)^* \tag{46}$$

for all objects A, B in the category. *Ribbon categories* are braided monoidal categories with the left duality ()* and a twist θ_A. *Modular tensor categories* are a kind of special ribbon categories, and they have simple objects $X_1, X_2, \cdots X_n$ with the fusion rule $X_i \otimes X_j \cong \oplus_{i,j=1}^n C_{ij}^k X_k$, fusion coefficients C_{ij}^k to be natural numbers (or zero) and an invertible symmetric s-matrix by its matrix entries $s_{ij} = Tr(\sigma_{X_i,X_j} \circ \sigma_{X_j,X_i})$. In the first footnote, *dagger ribbon categories*. i.e., ribbon categories with the dagger operation ()† coherently preserving the ribbon structure, $\sigma_{A,B}^\dagger = \sigma_{A,B}^{-1}$, $\theta_A^\dagger = \theta_A^{-1}$ and

$$\eta_A^\dagger = \epsilon_A \circ \sigma_{A,A^*} \circ (\theta_A \otimes Id_{A^*}), \quad \epsilon_A^\dagger = (Id_{A^*} \otimes \theta_A^{-1}) \circ \sigma_{A^*,A}^{-1} \circ \eta_A \tag{47}$$

have been proposed to be a underlying mathematical framework for *categorical quantum physics and information*. With the convention by Abramsky and Coecke for strongly compact closed categories, *dagger ribbon categories* can be also called strongly ribbon categories, see Paquette's unpublished thesis [12] [41].

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[12]In this unpublished reference (informed by Coecke in his feedback to Zhang on the first web version of the present manuscript), dagger ribbon categories are involved but all of those key proposals in the first footnote and last section are not realized and made.

Since Hilbert space is an object with positive definite forms between two linearly bounded operators, dagger ribbon categories in the proposal for *categorical quantum physics and information* have to be equipped with positive definite forms between two morphisms, i.e., positive dagger ribbon categories also called unitary Hermitian ribbon categories in Turaev's book [28]. But for dagger ribbon categories with some specific fusion rules, one can not achieve both positive definite forms and unitary braidings due to compatibility conditions between them, see interesting examples in Rowell's papers [42, 43]. The manuscript [44] to be an extension of the proposal in the first footnote and last section will completely exploit Turaev's notations on Hermitian ribbon categories.

Besides categories introduced as above, the d-dimensional *cobordism category* **Cob[d]** to define TQFT by Atiyah [20], has as its objects smooth manifolds of dimension d, and as its morphisms, smooth manifolds M^{d+1} of dimension $d + 1$ with a partition of the boundary, ∂M^{d+1}, into two collections of d-manifolds that we denote by $L(M^{d+1})$ and $R(M^{d+1})$. We regard M^{d+1} as a morphism from $L(M^{d+1})$ to $R(M^{d+1})$, $M^{d+1} : L(M^{d+1}) \implies R(M^{d+1})$. These categories **Cob[d]** are highly significant for quantum physics & information, especially **Cob[0]** is directly related to Dirac notations of quantum mechanics and to the Brauer category [13] or TL category [10].

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